FLOW OF A CONDUCTING GAS JET BEYOND THE NOZZLE OUTLET OF AN ELECTROMAGNETIC ACCELERATOR

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A solution is given for the problem of the motion of a conducting gas beyond the outlet of an accelerator. The form of the jet is found as well as the distribution of all jet parameters. The problem is solved assuming that the flow is plane, that there are no Hall currents, and that the velocity increase in the jet is small compared with the magnitude of the velocity at the exit of the accelerator channel.

1. We shall consider a jet of conducting gas flowing out of an accelerator. Since the jet flows into a vacuum a zone of rarefaction should arise at the jet boundary. But in view of the fact that the gas velocity is large (from $\sim 5 \cdot 10^6$ to 10^7 cm/sec), the M number will also be very large (M ~ 25 or more). The angle of inclination of the first characteristic of the rarefied zone relative to the jet axis will be at the most of the order of a few degrees. Thus at a distance equal to several times the height of the accelerator channel beyond its outlet, the core of the stream will occupy practically the entire height of the jet, and the zone of rarefaction will be only a narrow boundary layer. Thus in what follows we shall neglect the gasdynamic expansion of the jet and consider a flow pattern which is the result of the action of electromagnetic forces exclusively. This means that we shall neglect the pressure gradient in the equations in comparison with the electromagnetic forces.

We shall make the further assumption that the accelerator channel has a rectangular cross section which is wide enough for the motion to be treated as plane. Usually the thickness of the electrodes is small compared with the channel height. Thus, for simplicity, we shall consider the electrodes to be plates of zero thickness in what follows. When this simplification is made there may be a singular point at the end of the electrode in the mathematical solution for the electric potential. However, since the value of the electric potential does not enter into either the equations or the boundary conditions, it is quite permissible to make this idealization.

We shall let the x axis lie in the direction of the jet axis, and the y axis in the direction of the channel height. Let the coordinate origin be situated at the accelerator outlet on the lower electrode. On the assumptions made, the motion of the conducting gas jet will be described by the following system of equations:

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{4\pi\sigma}{c^2}\left(\frac{\partial UH}{\partial x} + \frac{\partial vH}{\partial y}\right),$$

$$\frac{\partial \rho U}{\partial x} + \frac{\partial \rho v}{\partial y} = 0,$$

$$\rho U \frac{\partial U}{\partial x} + \rho v \frac{\partial U}{\partial y} + \frac{1}{4\pi} H \frac{\partial H}{\partial x} = 0,$$

$$\rho U \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{1}{4\pi} H \frac{\partial H}{\partial y} = 0. \qquad (1.1)$$

Here ρ is the gas density, U and v are the x and y components of the velocity, H is the magnetic field strength, σ is the conductivity of the gas which will be assumed constant, and c is the velocity of light.

We shall now formulate the boundary conditions for this system. Strictly speaking the gas motion inside the channel of the accelerator cannot be treated independently of the motion outside the channel, since the presence of a conducting jet outside the channel causes a distortion of the lines of electric current at the end of the channel. However, in the case when there are no Hall currents and the channel is fairly long, the x component of the current in the channel may be neglected compared with the y component, and in the first approximation we may take the electric current lines at the end of the channel to be straight and parallel to the y axis. When this assumption is made, the motion in the channel may be calculated first of all, and then the following boundary conditions for the density and velocity may be assumed in solving the system (1.1):

$$U = U(y), \quad v = v(y), \quad \rho = \rho(y) \quad \text{for } x = 0.$$

Since no currents flow outside the jet, the condition H = 0 should hold at the jet boundaries. It is clear from Eqs. (1.1) that U = const and v = const for H = 0. Thus the jet boundaries will be straight lines, i. e., H = 0 for y = −kx and $y = y_0 + kx$, where y_0 is the channel height, and k is a constant to be determined in the course of solving the problem.

At some distance from the accelerator exit the conductivity decreases sharply due to cooling of the gas as a consequence of radiation. Thus it may be assumed that beyond a certain cross section of the jet, currents no longer flow. The second boundary condition for H is then H = 0 for $x = x_0$. Finally the assumption that the lines of electric current inside the channel close to its end are parallel to the y axis leads to the conclusion that $H = H_0$ = const for x = 0. The value of this constant may be determined after solving the problem, starting, for example, from the known potential difference on the electrodes or from the magnitude of the total current (depending on the particular specific conditions of the problem).

In considering the motion in the jet we shall assume that H_0 is a given quantity in what follows. In addition we shall, for simplicity, assume that the density is constant over the exit cross section of the accelerator, and that the velocity is a constant quantity and in the direction of the x axis. Thus the boundary conditions assume the form

$$U = u_0, \quad v = 0, \quad \rho = \rho_0, \quad H = H_0 \quad \text{for } x = 0,$$

$$H = 0 \text{ for } y = -kx,$$
$$y = y_0 + kx \text{ and for } x = x_0. \qquad (1.2)$$

Here μ_0, ρ_0, H_0 are given constants, and k is to be determined from the condition that the velocity be in the same direction as the boundary at the jet boundary.

The system of equations (1.1) may be simplified if we take account of the following properties of the flow in the case under consideration. If the accelerator channel is sufficiently long (very short channels, the so-called end type accelerators, will not be treated here), then the fraction of the current flowing outside the accelerator is small compared with the currents flowing inside the accelerator. Thus the change in jet velocity outside the accelerator is small compared with the magnitude of the velocity at the accelerator exit. The angle of broadening of the jet is also small, and so if we are considering motion at distances from the exit which are of the order of several times the channel height, then the change in gas density will also be small in this zone. We note, moreover, that since the dimensions of the zone in which the main current in the jet flows in the directions of the x and y axes are of the same order, the derivatives with respect to x and y should also be of the same order.

We set $U = u_0 + u$, where $u \ll u_0$. Then taking into consideration the assumptions made above concerning the flow pattern in the present case, we may linearize the equations of motion and induction:

$$\rho_0 u_0 \frac{\partial u}{\partial x} + \frac{H}{4\pi}\frac{\partial H}{\partial x} = 0, \quad \rho_0 u_0 \frac{\partial v}{\partial x} + \frac{H}{4\pi}\frac{\partial H}{\partial y} = 0, \quad (1.3)$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{4\pi\sigma u_0}{c^2}\frac{\partial H}{\partial x}. \qquad (1.4)$$

It follows from (1.3) that the values of the velocity components may be found in this approximation without making use of the continuity equations. After the velocity components have been found from the continuity equations, the density distribution in the jet may be found numerically.

The field strength H may be determined from Eq. (1.4) independently of u and v. With H as a function of x and y, the values of u and v themselves may easily be found by integrating (1.3) allowing for the boundary conditions (1.2):

$$u = \frac{1}{8\pi\rho_0 u_0}[H_0^2 - H^2(x, y)],$$
$$v = -\frac{1}{8\pi\rho_0 u_0}\int_0^x \frac{\partial H^2}{\partial y}dx. \qquad (1.5)$$

Thus the solution of the problem is reduced to finding H from Eq. (1.4) and the boundary conditions (1.2).

With the same degree of accuracy we may, in solving Eq. (1.4), require that the boundary condition $H = 0$ be fulfilled not on the lines $y = -kx$ and $y = y_0 + kx$, but on $y = 0$ and $y = y_0$, straight lines parallel to the jet axis.

We now look for a solution of Eq. (1.4) in the form $H = f_1(x) f_2(y)$. In order to determine f_1 and f_2 we have the equations

$$d^2 f_1 / dx^2 - (4\pi\sigma u_0 / c^2)\, df_1 / dx - C^2 f_1 = 0,$$
$$d^2 f_2 / dy^2 + C^2 f_2 = 0.$$

Integrating these equations we find the particular solution which goes to zero for $y = 0$, $y = y_0$, and $x = x_0$,

$$H_n = C_n \exp(A\pi x / y_0) \times$$
$$\times \operatorname{sh}\{\sqrt{A^2 + n^2}[\pi(x_0 - x)/y_0]\} \sin n\pi y / y_0,$$
$$A = 2\sigma u_0 y_0 / c^2.$$

Summing these particular solutions (1.6) and determining C_n from the condition that $H = H_0$ for $x = 0$, we find the required solution of the equation

$$H = \frac{4H_0}{\pi}\left(\exp\frac{A\pi x}{y_0}\right)\sum_{n=0}^{\infty}\frac{1}{2n+1} \times$$
$$\times \frac{\operatorname{sh}[\sqrt{A^2 + (2n+1)^2}\,\pi(x_0 - x)/y_0]}{\operatorname{sh}[\sqrt{A^2 + (2n+1)^2}\,\pi x_0 / y_0]} \sin\frac{(2n+1)\pi y}{y_0}. \qquad (1.7)$$

After the values of H(x, y), u(x, y), and v(x, y) have been calculated from formulas (1.5) and (1.7) the density ρ may be determined from the continuity equation.

The continuity equation is linearized, and the relation obtained is integrated taking the boundary conditions into account together with formulas (1.5) to give

$$\frac{\partial\rho}{\partial x} = -\frac{\rho_0}{u_0}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),$$
$$\rho = \rho_0 + \frac{1}{8\pi u_0^2}\left[H^2 - H_0^2 + \int_0^x dx \int_0^x \frac{\partial^2 H^2}{\partial y^2}dx\right]. \qquad (1.8)$$

Formulas (1.5), (1.7), and (1.8) give a complete solution of the problem of the flow of a conducting gas beyond the outlet of an accelerator.

2. In passing to a more detailed analysis we note the important fact that for the values of x_0 treated ($x_0/y_0 > 3$–4) the distribution of H, and consequently of all the other quantities, is practically independent of x_0. Actually the factors in (1.7), dependent on x_0, have the form

$$\operatorname{sh}\frac{k_1(x_0 - x)}{y_0}\operatorname{csch}\frac{k_1 x_0}{y_0} = \operatorname{sh}\frac{k_1 x}{y_0}\operatorname{cth}\frac{k_1 x_0}{y_0} - \operatorname{ch}\frac{k_1 x}{y_0}.$$

Here $k_1 > 3$ for $n = 0$ and increases with n. However, for $k_1 > 3$ and $x/y_0 > 3$ we have $\operatorname{th} k_1 x_0 / y_0 = 1$ with an accuracy to 10^{-8}. Taking this into account expression (1.7) may be simplified by calculating H for x_0:

$$H = \frac{4H_0}{\pi}\exp\frac{A\pi x}{y_0}\sum_{n=0}^{\infty}\frac{1}{2n+1} \times$$
$$\times \exp\frac{-\sqrt{A^2 + (2n+1)^2}\,\pi x}{y_0}\sin\frac{(2n+1)\pi y}{y_0}. \qquad (2.1)$$

Further simplification of this formula results if we take into account the fact that for characteristic values of the parameters, i. e., for $\sigma \sim 10^{13}\ \text{sec}^{-1}$, $u_0 \sim 5\cdot 10^6$ cm/sec, $y_0 \sim 3$ cm, it follows that $A \sim 0.3$, and consequently A^2 may be neglected in comparison with $(2n + 1)^2$ for any n. In this case the series in (2.1) may be summed (see, for example, [1]) and the following expression for H is obtained:

$$H = \frac{2H_0}{\pi}\left(\exp\frac{A\pi x}{y_0}\right)\operatorname{arc\,tg}\left(\sin\frac{\pi y}{y_0}\operatorname{csch}\frac{\pi x}{y_0}\right). \quad (2.2)$$

Figure 1 shows the lines H = const (lines of electric current), calculated from formula (2.2). The solid curves correspond to A = 0, and the dashed ones to A = 0.1. We see from Fig. 1 that at a distance of two units from the accelerator outlet the magnetic field falls off practically to zero ($H/H_0 \sim 0.01$), while for $A \neq 0$ the magnetic field decreases somewhat more slowly. Setting (2.2) into (1.5), we have

$$v = -\frac{H_0^2\cos(\pi y/y_0)}{\pi^2\rho_0 u_0 y_0}\int_0^x\left(\exp\frac{2A\pi x}{y_0}\right)\times$$
$$\times\left[\operatorname{arc\,tg}\frac{\sin(\pi y/y_0)}{\operatorname{sh}(\pi x/y_0)}\right]\frac{\operatorname{sh}(\pi x/y_0)\,dx}{\sin^2(\pi y/y_0)+\operatorname{sh}^2(\pi x/y_0)}. \quad (2.3)$$

Figures 2 and 3 show profiles of the velocity v calculated from (2.3) for A = 0.1. Figure 2 presents

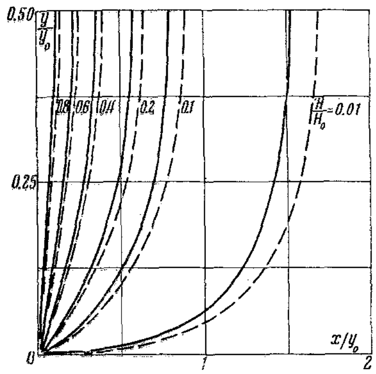

Fig. 1

the function $v(x/y_0)$ for a series of values of $\pi y/y_0 = y^*$, while Fig. 3 presents the function $v(y/y_0)$ for various values $\pi x/x_0 = x^*$. Values of $v^* = -(100\rho_0 u_0/H_0^2)v$ are plotted on the ordinate axis in Figs. 2 and 3.

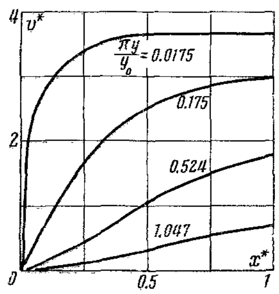

Fig. 2

It is clear from Fig. 3 that the absolute magnitude of the y velocity component increases monotonically from the center of the jet to the boundaries. The maximum absolute value of v may be obtained from (2.3) by the limiting transition for $y \to 0$. It is independent of the value of the parameter A. Actually for $y/y_0 \ll 1$ we make the substitution $\pi y/y_0 = \delta^3$ and split the interval of integration in (2.3) into two parts: from zero to x_1 and from x_1 to x, choosing x_1 such that the condition $\pi x_1/y_0 \sim \delta$ is fulfilled. We may then write

$$v = -\frac{H_0^2}{\pi^2\rho_0 u_0 y_0}\left\{\int_0^{x_1}[1 + AO(\delta)]\left[\operatorname{arc\,tg}\frac{\sin(\pi y/y_0)}{\operatorname{sh}(\pi x/y_0)}\right]\times\right.$$
$$\times\frac{\operatorname{sh}(\pi x/y_0)\,dx}{\sin^2(\pi y/y_0)+\operatorname{sh}^2(\pi x/y_0)}+$$
$$\left.+\int_{x_1}^{x}\left(\exp\frac{2A\pi x}{y_0}\right)\frac{\delta^3[1+O(\delta)]}{\operatorname{sh}^2(\pi x/y_0)}\,dx\right\}.$$

For $\delta \to 0$ the first integral tends to a quantity independent of A, and the second to zero. Thus for any A,

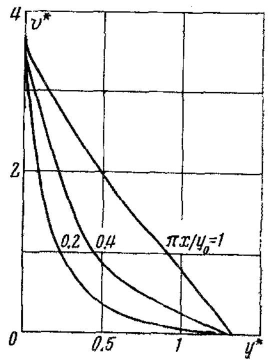

Fig. 3

within the approximation used in solving the problem, the jet cone angle α is the same. Its absolute magnitude is

$$\alpha = \operatorname{arc\,tg}|v/u_0| = \operatorname{arc\,tg}(0.037\,H_0^2/\rho_0 u_0^2). \quad (2.4)$$

Figure 4 shows the distribution of the increase in the x component of velocity calculated from formulas

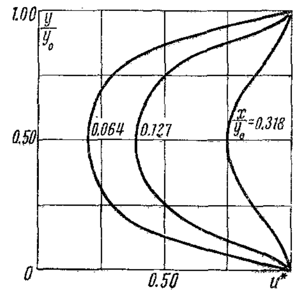

Fig. 4

(1.5) and (2.2) for A = 0.1 at different cross sections of the jet. The unit on the abscissa axis is the quantity $u^* = (8\pi\rho_0 u_0/H_0^2)u$, while that of the ordinate axis is the ratio y/y_0. The profile of u is markedly nonuniform at a distance of from two to three times the channel height from the exit.

3. It is of interest to consider the flow beyond the outlet of a channel with three electrodes: two external electrodes at the same potential and an internal electrode at a different potential. The flow in the jet emerging from such a channel should be similar to the flow beyond the outlet of a coaxial accelerator. In this case we shall let the coordinate origin be situated at the end of the center electrode. Let the coordinates of the ends of the outer electrodes be $\pm y_0$, respectively.

The distribution of all quantities may be found everywhere, with the exception of the zone behind the middle electrode, from the same formulas as before (formulas (1.5), (2.2), (2.3), (2.4)). Since the velocity v has different signs on either side of the middle electrode, this should lead to the formation of shock waves leaving the end of the middle electrode.

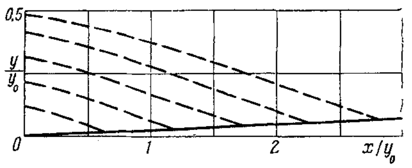

Fig. 5

We shall carry out some calculations for these compression shocks and the flow behind them in the first approximation, assuming that the shocks are strong enough for the pressure in front of them to be neglected in comparison with the pressure behind them. Since the M number for the stream in front of the shock is very large, the angle of inclination β of the shock relative to the x axis is small, and consequently the height of the zone behind the shock is small compared with the width of the channel. Taking this into account, as well as the fact that the shock wave arises as it were from nonuniform flow around a wedge (it follows from symmetry that the x axis is a streamline of zero curvature), we may neglect, within the degree of accuracy to which we are working here, the pressure variation across the zone behind the shock wave. Quantities in front of the shock wave will be denoted by letters without an index, while letters with the superscript ° will be used for quantities behind the shock wave. On the assumptions already made we may write [2]

$$p^\circ = \frac{2}{\varkappa+1}\rho u_n^2, \quad \rho^\circ = \frac{\varkappa+1}{\varkappa-1}\rho,$$
$$u_n^\circ = \frac{\varkappa-1}{\varkappa+1}u_n, \quad u_\tau^\circ = u_\tau, \tag{3.1}$$

where u_n, u_τ are the velocity components normal and tangential to the shock wave. They may be expressed in terms of U, v and β, the angle of inclination of the shock to the x axis, as follows:

$$u_\tau = U\cos\beta + v\sin\beta \approx u_0 + u,$$
$$u_n = -U\sin\beta + v\cos\beta \approx -u_0\beta + v. \tag{3.2}$$

Here the second halves of the equations allow for the fact that $\beta \ll 1$, $v/u_0 \ll 1$, and $u/u_0 \ll 1$. The coordinate of the compression shock y_s may be calculated from considerations of jet flow behind a shock. Let the gas in a jet flow which has intersected a shock wave at a cross section of coordinate x_* have a density $\rho^0(x, x_*)$ at the cross section with coordinate x which is under consideration. Then to the first approximation the rate of gas flow in this jet stream at the cross section in question is $dq = u_0\rho^0(x, x_*)dy$. On the other hand if we consider this jet at the cross section x_* we find that

$$dq = -\rho(x_*)\,u_n(x_*)\,dx_*,$$
$$u_0\rho^\circ(x, x_*)\,dy = -\rho(x_*)\,u_n(x_*)\,dx_*. \tag{3.3}$$

The first of these equations allows for the fact that $\cos\beta \approx 1$.

Assuming that the flow behind the shock is adiabatic and taking (3.1) into account we find that

$$\frac{\rho(x_*)}{\rho^\circ(x, x_*)} = \frac{\varkappa-1}{\varkappa+1}\frac{\rho^\circ(x_*, x_*)}{\rho^\circ(x, x_*)} = \frac{\varkappa-1}{\varkappa+1}\left[\frac{p^\circ(x_*)}{p^\circ(x)}\right]^{1/\varkappa}. \tag{3.4}$$

Allowing for (3.4) we have from (3.3)

$$y_s = -\frac{\varkappa-1}{\varkappa+1}\int_0^x \frac{u_n(x_*)}{u_0}\left[\frac{p^\circ(x_*)}{p^\circ(x)}\right]^{1/\varkappa} dx =$$
$$= -\frac{\varkappa-1}{\varkappa+1}\int_0^x \frac{[\beta(x_*) - v(x_*)/u_0]^{(\varkappa+2)/\varkappa}}{[\beta(x) - v(x)/u_0]^{2/\varkappa}}\, dx_*. \tag{3.5}$$

Differentiating (3.5) with respect to x and remembering that $\beta = y_s'$, $\beta' = y_s''$, we obtain the differential equation for determining y_s:

$$\frac{2}{\varkappa} y_c y_c'' - \frac{1}{\varkappa+1}\left[2y_c' + (\varkappa-1)\frac{v}{u_0}\right]\times$$
$$\times\left(y_s' - \frac{v}{u_0}\right) - \frac{2}{\varkappa} y_s \frac{v'}{u_0} = 0. \tag{3.6}$$

The boundary conditions in this case are

$$y_s = 0, \quad y_s' = \beta_0 = -0.5(\varkappa-1)v_0/u_0 \quad \text{for } x = 0. \tag{3.7}$$

The second boundary condition expresses the fact that for x = 0 the velocity v° behind the shock is zero. We thus obtain (3.7) from (3.1) and (3.2). Integrating (3.6) with the boundary conditions (3.7) we find the form of the compression shock. The solid line in Fig. 5 shows the form of the compression shock (in the upper half of the channel), calculated for $\varkappa = 5/3$ and $H_0^2/\rho_0 u_0^2 = 2.5$ (this corresponds to the case when the velocity outside the accelerator increases by about 10% of its value at the accelerator cutoff). The dashed line in the same figure shows the stream line in the channel calculated from the equation $dy/dx = v/u_0$, where v is given by formula (2.3). The stream lines were calculated for the parameter value A = 0.1.

The plasma flow pattern beyond the accelerator cutoff obtained above is well-confirmed by experimental data, at least qualitatively.

Since the gas behind the compression shock has a higher temperature than in the main core of the stream, the region behind compression shocks radiates much more brightly. Usually in experiments the boundaries of a brightly glowing zone are clearly evident behind the central electrode. These boundaries are compression shocks.

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